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Quasiclassical descriptions of quantum systems based on coherent states: product formulae

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Abstract. If the state space of a quantum system is finite-dimensional it may be considered as carrier space of an irreducible unitary representation of some compact Lie group $\mathcal{G}$. In this case the set of operators may be put in one-to-one correspondence with a class of functions defined on the group parameters (Q representation). For these functions a binary relation is derived that corresponds to the product of two operators. The general formalism is applied to $\mathcal{G} = \mathrm{SU}(2)$ (spin coherent states) and this result is used to derive a product formula for the non-compact Heisenberg–Weyl group (coherent oscillator states).

1. Introduction

To describe the properties of a quantum system one needs (i) a star-algebra, i.e. a non-commutative algebra $\mathcal{A}$ that admits an anti-automorphism of order two ($a^{**} = a$ for all $a \in \mathcal{A}$); and (ii) a linear mapping C from $\mathcal{A}$ (or a subalgebra) to the complex numbers [1]. Part of the elements of $\mathcal{A}$, euphemistically called observables, correspond to physically measurable quantities; those elements that may be represented as $w = aa^*$ and satisfy $C(w) = 1$ characterize the state of the system; the remaining elements are introduced to obtain a mathematically well-defined theory. The states form a convex set where the extremal elements, called pure states, satisfy $w^2 = w$. Products of elements occur not only in the equations that specify the state symbols but also in the expectation values $C(wa)$. The non-commutativity of the multiplication essential for describing changes of the state of the system (e.g. the evolution in time), no matter whether the transformation is defined globally ($w \to uwu^*$) or locally ($w \to wx - xw$).

In the traditional formulation of quantum mechanics the observables $a, b, \ldots$ are identified with operators $\hat{A}, \hat{B}, \ldots$ in a Hilbert space and the anti-automorphism $a \to a^*$ with the mapping $\hat{A} \to \hat{A}^\dagger$. The state symbol is then the density operator $\hat{W}$ and the linear functional that assigns numbers to the operators is the trace operation ($C(a) = \mathrm{Tr}\,\hat{A}$). The product of two operators is defined by the successive action on the elements of the Hilbert space; in practical calculations this is reduced to the multiplication of matrices. Although this mathematical construction has been used in research and education almost exclusively, alternative formulations, were proposed soon after the discovery of quantum mechanics. Weyl [2] intoduced a correspondence rule for operators (polynomials in the operators $\hat{x}$ and $\hat{p}$) and phase space functions (polynomials in the real variables x and p) while Wigner [3] studied the properties of the distribution function W corresponding to a density operator $\hat{W}$. In the resulting formulation of quantum mechanics every observable a is represented by an ordinary function A and the involutive anti-automorphism $a \rightarrow a^*$ appears as complex conjugation $A \rightarrow A^*$. The non-commutative composition law that corresponds to the product of two operators was derived by Groenewold [4]; the commutator obtained from this 'product' is now known as the Moyal bracket [5]. The complex number $C(a)$ turns out to be proportional to $\bar{A}$, the average of the function A obtained by integration over its domain. This phase-space formulation of quantum mechanics is now well documented (see, e.g., [6–8]) but still comparatively rarely used, probably because most quantum mechanical problems are by no means easier to solve in this form than in the usual Hilbert space formulation. However, this description of quantum mechanics has an appealing similarity with classical statistical mechanics and is therefore extremely useful in discussing quantum corrections to classical results. In the limit $\hbar \downarrow 0$ all observables commute while $\hbar$ times the Moyal bracket approaches the Poisson bracket of classical mechanics.

Another quasiclassical description of quantum systems is related to coherent states [9]. For a spinless particle in one dimension these states can be labelled by two parameters, say p and x, both varying over the whole real line. Expectation values for these states exist for a large class of operators including bounded operators and polnomials in $\hat{x}$ and $\hat{p}$. This fact can be used to assign to each operator $\hat{A}$ in this class a function $A(p, x)$, the so-called Q representative of $\hat{A}$. (For a large variety of operators there also exists a second function $A'(p, q)$, called the P representative which is uniquely related to $A(p, q)$, but this is not used in the following discussion.) This phase-space representation of observables was originally studied for the harmonic oscillator [10] and modes of the electromagnetic field (hence the name 'coherent states' [11]). Later on the concept of coherent states and the related Q representation of operators was also introduced for spin systems [12–14] and finally generalized to other systems whose state space carries an irreducible representation of a Lie group [15,16]. In all these examples the relationship between the Q representatives of $\hat{A}$ and $\hat{A}^\dagger$ is given by complex conjugation as was the case in the Wigner–Weyl formalism; likewise the trace operation is always transformed into an integral over the domain of the Q representatives. To obtain a closed mathematical structure that is fully equivalent to the usual Hilbert space formulation of quantum mechanics one also needs a composition law for the Q representatives which replaces the product of operators. In view of the extensive discussion of coherent states (see, e.g., [9] for further references) it is surprising that general formulae of this kind are apparently still lacking. What can be found in the literature are only formulae for the trace of the product of two arbitrary operators (here the P representatives enter beside the Q representatives [9]) or formulae where one of the factors belongs to the Lie algebra [17–19].

In this paper a general product formula is derived for systems related to compact semi-simple Lie groups. To make the presentation self-contained some facts from the representation theory of these groups are reviewed in section 2 before the formula is derived in section 3. In section 4 the general scheme is applied to SU(2) (coherent spin states) and the corresponding formulae for the non-compact Heisenberg–Weyl group H(4) (coherent oscillator states) are obtained by contraction in section 5. Finally our conclusions are summarized in section 6.

2. Basic concepts and notation

In this section some elements of the representation theory of compact semi-simple Lie groups are briefly reviewed; a detailed discussion is found in many textbooks on group theory (see, e.g., [20–22]).

Let $\mathcal{G}$ be a compact group with elements g. The complex-valued functions defined on $\mathcal{G}$ form a Hilbert space $L^2(\mathcal{G})$ which is separable for all groups of physical interest. The scalar product in $L^2(\mathcal{G})$ is defined by means of the Haar measure $\mathrm{d}\mu(g)$ which is uniquely determined by $\mathcal{G}$ up to a normalization constant. If $\mathcal{G}$ is a Lie group the elements g may be parametrized by a set of real coordinates $\eta = (\eta_1, \eta_2, \ldots)$. There always exist global coordinates which label the group elements uniquely except for a set of measure zero that may be neglected in the following considerations. Accordingly functions on $\mathcal{G}$ may be identified with functions of the global coordinates ($f(g(\eta)) = F(\eta)$). In the Hilbert space of these functions it is sufficient to consider analytical functions because these are dense in $L^2(\mathcal{G}, \eta)$. The Haar measure becomes a weighted integral over the coordinates, the weight function $\mu(\eta)$ being uniquely fixed by the condition $\int \mu(\eta)\,\mathrm{d}\eta = 1$. In $L^2(\mathcal{G}, \eta)$ the scalar product and the norm are defined by

$$\langle A, B\rangle = \int A^*(\eta)B(\eta)\mu(\eta)\,\mathrm{d}\eta \qquad \|A\|^2 = \langle A, A\rangle. \tag{1}$$

For group elements in the neighbourhood of the unit element it is often more convenient to use other parameters than the global coordinates η, namely canonical coordinates ξ for which $g(\xi)^{-1} = g(-\xi)$.

A unitary operator $\check{U}^{\mathrm{R}}(g_1)$ in $L^2(\mathcal{G})$ is assigned to each group element $g_1 \in \mathcal{G}$ through $[\check{U}^{\mathrm{R}}(g_1)f](g) = f(gg_1)$ (right-regular representation of $\mathcal{G}$). If $g_1 = g(\xi)$ and $g = g(\eta)$ then

$$[\check{U}^{\mathrm{R}}(\xi)F](\eta) = f(g(\eta)g(\xi)) = f(g(\tilde{\eta})) = F(\tilde{\eta}) \tag{2}$$

$$\tilde{\eta}_l = \eta_l + \sum_k \Phi_{lk}(\eta)\xi_k + \mathrm{O}(\xi^2). \tag{3}$$

For given coordinates ξ and η the square matrix $\Phi(\eta)$ is determined by the multiplication law of $\mathcal{G}$. For analytic functions

$$[\check{U}^{\mathrm{R}}(\xi)F](\eta) = F(\eta) - \mathrm{i}\sum_k \xi_k[\check{K}_k^{\mathrm{R}}F](\eta) + \mathrm{O}(\xi^2) \tag{4}$$

where the operators

$$\check{K}_k^{\mathrm{R}} = \mathrm{i}\sum_l \Phi_{lk}(\eta)\frac{\partial}{\partial\eta_l} \tag{5}$$

are self-adjoint because the operators $\check{U}^{\mathrm{R}}(\xi)$ are unitary. As the commutation relations of these operators fix the multiplication law of $\mathcal{G}$ in the neighbourhood of the unit element the set of complex linear combinations of the operators (5), endowed with the commutator as Lie product, is isomorphic to the complex Lie algebra g of the group $\mathcal{G}$. If this algebra is semisimple it is possible to choose as basis of g operators $\check{H}_1^{\mathrm{R}}, \ldots, \check{H}_r^{\mathrm{R}}, \check{E}_\alpha^{\mathrm{R}}, \check{E}_\beta^{\mathrm{R}}, \ldots, \check{E}_{-\alpha}^{\mathrm{R}}, \check{E}_{-\beta}^{\mathrm{R}}, \ldots$ for which

$$\check{H}_i^{\mathrm{R}\dagger} = \check{H}_i^{\mathrm{R}} \qquad \check{E}_\alpha^{\mathrm{R}\dagger} = \check{E}_{-\alpha}^{\mathrm{R}} \tag{6}$$

since the expansion coefficients with respect to the operators $\check{K}_k^{\mathrm{R}}$ are real for $\check{H}_i^{\mathrm{R}}$ and complex conjugate for $\check{E}_{\pm\alpha}^{\mathrm{R}}$, and which satisfy the following canonical commutation rules.

$$\begin{aligned}
&[\check{H}_i^{\mathrm{R}}, \check{H}_j^{\mathrm{R}}] = 0 \\
&[\check{H}_i^{\mathrm{R}}, \check{E}_\alpha^{\mathrm{R}}] = \alpha_i \check{E}_\alpha^{\mathrm{R}} \\
&[\check{E}_\alpha^{\mathrm{R}}, \check{E}_{-\alpha}^{\mathrm{R}}] = \sum_i \alpha_i \check{H}_i^{\mathrm{R}} \\
&[\check{E}_\alpha^{\mathrm{R}}, \check{E}_\beta^{\mathrm{R}}] = N_{\alpha,\beta} \check{E}_{\alpha+\beta}^{\mathrm{R}} \\
&N_{\alpha,\beta}^2 = \frac{n(m+1)}{2} \sum_i \alpha_i^2.
\end{aligned} \tag{7}$$

In these equations the root vectors $\alpha = (\alpha_1, \ldots, \alpha_r)$ are assumed to be normalized according to

$$\sum_\alpha \alpha_i \alpha_j = \delta_{ij} \tag{8}$$

and the integers n and m are uniquely determined by the pair of root vectors α and β; if $\alpha + \beta$ is not a root vector $N_{\alpha,\beta} = 0$. In general there exist infinitely many linear combinations of the operators (5) that satisfy equations (6) and (7); in the following we consider one fixed set of such operators. In addition to these operators it is also interesting to consider multiple products of the non-commuting operators $\check{K}_k^{\mathrm{R}}$ and linear combinations thereof, especially those that commute with all operators of this form. All these commuting polynomials may be generated from r polynomials of the lowest order, the so-called Casimir operators $\check{C}_1^{\mathrm{R}}, \ldots, \check{C}_r^{\mathrm{R}}$.

By complete analogy with the right-regular representation a left-regular representation may be introduced through $[\check{U}^{\mathrm{L}}(g_1)f](g) = f(g_1^{-1}g)$. To simplify the calculation of the corresponding differential operators $\check{K}_k^{\mathrm{L}}$ it is convenient to introduce in $L^2(\mathcal{G})$ an involution $f \to f^\#$ through the definition $f^\#(g) = f(g^{-1})^*$. This implies $\check{U}^{\mathrm{L}}(g)\, f^\# = [\check{U}^{\mathrm{R}}(g)\, f]^\#$ and entails the following relations for the functions of the coordinates η and η^{-1}:

$$g(\eta)^{-1} = g(\eta^{-1}) \qquad F^\#(\eta) = F(\eta^{-1})^* \qquad \check{U}^{\mathrm{L}}(\xi)F^\# = [\check{U}^{\mathrm{R}}(\xi)F]^\#. \tag{9}$$

If the generators $\check{K}_k^{\mathrm{L}}$ are related to the unitary transformations $\check{U}^{\mathrm{L}}$ as the corresponding right operators were related in (4) then

$$\check{K}_k^{\mathrm{L}} = -\check{K}_k^{\mathrm{R}\#} = \mathrm{i} \sum_l \Phi_{lk}(\eta^{-1}) \frac{\partial}{\partial(\eta^{-1})_l}. \tag{10}$$

Here $\check{K}^\#$ means that (i) in $\check{K}$ the coordinates η_l are replaced by the coordinates $(\eta)^{-1}$ in all functions and differential operators; and (ii) the complex conjugate of this expression is taken. Although the operators (10) differ from the operators (5) their definition entails that they obey the same commutation rules. The operators $\check{H}_1^{\mathrm{L}}, \ldots, \check{E}_\alpha^{\mathrm{L}}, \ldots$ are related to the operators $\check{H}_1^{\mathrm{R}}, \ldots, \check{E}_\alpha^{\mathrm{R}}, \ldots$ by

$$\check{H}_i^{\mathrm{L}} = -\check{H}_i^{\mathrm{R}\#} \qquad \check{E}_\alpha^{\mathrm{L}} = -\check{E}_{-\alpha}^{\mathrm{R}\#} \tag{11}$$

and therefore also related to each other by equations (7). All operators belonging to the left-regular representation commute with those derived from the right-regular representation and the Casimir operators may be chosen in such a way that

$$\check{C}_i^{\mathrm{L}} = \check{C}_i^{\mathrm{R}} = \check{C}_i = \check{C}_i^{\dagger}. \tag{12}$$

Since the self-adjoint operators $\check{H}_i^{\mathrm{L}}$, $\check{H}_i^{\mathrm{R}}$ and $\check{C}_i$, commute with each other it is possible to find common eigenfunctions. The eigenvalue equations

$$\check{C}_i T^{\Lambda} = \Lambda_i T^{\Lambda} \tag{13}$$

have n_Λ^2 linearly independent solutions where n_Λ is an integer characterisic for the joint eigenvalue $\Lambda = (\Lambda_1, \ldots, \Lambda_r)$. The space spanned by these functions is invariant under the operators of both the left- and the right-regular representation. The remaining eigenvalue equations are

$$\check{H}_i^{\mathrm{L}} T^{\Lambda}_{\lambda',\lambda''} = \lambda'_i T^{\Lambda}_{\lambda',\lambda''} \qquad \check{H}_i^{\mathrm{R}} T^{\Lambda}_{\lambda',\lambda''} = \lambda''_i T^{\Lambda}_{\lambda',\lambda''}. \tag{14}$$

The joint eigenvalues $\lambda' = (\lambda_1, \ldots, \lambda_r)$ and λ'' are called weights and range over the same set of vectors. Equations (14) admit in general several linearly independent solutions but there exist weights λ^e such that the solution is non-degenerate for $\lambda' = -\lambda'' = \lambda^e$. One of these weights is selected by a convention and called the maximal weight; in the following this weight is denoted by λ. For the solution of (14) with $\lambda' = -\lambda'' = \lambda$

$$\check{E}_\alpha^{\mathrm{L}} T^{\Lambda}_{\lambda,-\lambda} = 0 = \check{E}_{-\alpha}^{\mathrm{R}} T^{\Lambda}_{\lambda,-\lambda} \tag{15}$$

for all 'positive' root vectors α. The eigenvalue equations determine the function $T^{\Lambda}_{\lambda,-\lambda}$ up to a factor that is fixed by the convention

$$T^{\Lambda}_{\lambda,-\lambda}(\eta_o) = 1 \qquad (g(\eta_0) = 1 - \text{ element of } \mathcal{G}). \tag{16}$$

Note that (16) entails

$$T^{\Lambda\#}_{\lambda,-\lambda} = T^{\Lambda}_{\lambda,-\lambda}. \tag{17}$$

Further solutions of the eigenvalue equations (13), (14) are obtained by applying a sufficient number of operators $\check{E}_{-\alpha}^{\mathrm{L}}$ to the function $T^{\Lambda}_{\lambda,-\lambda}$. Let

$$\pm p = \pm\alpha \pm \beta \pm \cdots \pm \gamma \tag{18}$$

be 'Gel'fand–Tsetlein patterns' [23,24] that label the operator products

$$\check{E}_{\pm p} = (\mp\check{E}_{\pm\gamma}) \ldots (\mp\check{E}_{\pm\beta})(\mp\check{E}_{\pm\alpha}). \tag{19}$$

It follows from the commutation relations (7) that $\check{E}_{-p}^{\mathrm{L}} T^{\Lambda}_{\lambda,-\lambda}$ is a solution of (14) belonging to the weights

$$\lambda' = \lambda - p = \lambda - \alpha - \beta - \ldots - \gamma \tag{20}$$

and $\lambda'' = -\lambda$, which is normalized to

$$\|\check{E}^{\mathrm{L}}_{-p} T^{\Lambda}_{\lambda,-\lambda}\|^2 = \|\check{E}^{\mathrm{R}}_{+p} T^{\Lambda}_{\lambda,-\lambda}\|^2 = (\lambda \mid p)\|\check{T}^{\Lambda}_{\lambda,-\lambda}\|^2. \tag{21}$$

The factors $(\lambda \mid p)$ may be calcuated using relations (7) and (15). From the commutation relations (7) and the eigenvalue equations (14) it follows that

$$(\lambda \mid p) \geq (\lambda, \alpha)(\lambda - \alpha, \beta) \ldots (\lambda - \alpha - \beta - \ldots, \gamma) \tag{22}$$

$$(\rho, \sigma) = \sum_i \rho_i \sigma_i. \tag{23}$$

By this procedure one obtains n_Λ orthogonal solutions of (13), (14) that may be labelled by a fixed set of patterns $\{-p\}$ including the pattern $p = 0$.

$$(\lambda \mid 0) = 1 \qquad \check{E}^{\mathrm{L}}_0 = -\check{E}^{\mathrm{R}}_0 = \check{1} \text{ (unit operator)}. \tag{24}$$

If the operators $\check{E}^{\mathrm{R}}_{+p'}$, $p' \in \{+p\}$, are applied to one of these functions n_Λ new orthogonal solutions of (13), (14) are generated. This finally yields n^2_Λ functions labelled by indices $p', p'' \in \{+p\}$.

$$D^{\Lambda *}_{p',p''} = -\omega(\lambda, p')\omega(\lambda, p'')^*(\lambda \mid p')^{-1/2}(\lambda \mid p'')^{-1/2} \check{E}^{\mathrm{L}}_{-p'} \check{E}^{\mathrm{R}}_{+p''} T^{\Lambda}_{\lambda,-\lambda}. \tag{25}$$

In (25) the numbers $\omega(\lambda, p)$ are phase factors that are fixed by conventions ($\omega(\lambda, 0) = 1$).

The functions (25) have the following properties.

(i) Under the involution they are either invariant or correlated in pairs

$$D^{\Lambda * \#}_{p',p''} = D^{\Lambda *}_{p'',p'}. \tag{26}$$

(ii) They are orthonormalized according to

$$\langle D^{\Lambda *}_{p',p''}, D^{N *}_{r',r''} \rangle = n_\Lambda^{-1} \delta_{\Lambda,N} \delta_{p',r'} \delta_{p'',r''}. \tag{27}$$

(iii) They engender an irreducible representation of the Lie algebra $\mathbf{g}$.

$$\check{K}^{\mathrm{L}}_k D^{\Lambda *}_{p',p''} = +\sum_p \left(\mathsf{K}^{\Lambda}_k\right)_{p,p'} D^{\Lambda *}_{p,p''}$$

$$\check{K}^{\mathrm{R}}_k D^{\Lambda *}_{p',p''} = -\sum_p \left(\mathsf{K}^{\Lambda}_k\right)^*_{p,p''} D^{\Lambda *}_{p',p}. \tag{28}$$

The representatives K^{Λ}_k of the operators $\check{K}^{\mathrm{L}}_k$ are Hermitian matrices from which the matrices H_i, E_α representing the operators $\check{H}^{\mathrm{L}}_i$, $\check{E}^{\mathrm{L}}_\alpha$ are obtained by linear combinations:

$$\check{H}^{\mathrm{L}}_j D^{\Lambda *}_{p',p''} = +\left(\mathsf{H}^{\Lambda}_k\right)_{p',p'} D^{\Lambda *}_{p',p''} = +(\lambda - p') D^{\Lambda *}_{p',p''}$$

$$\check{H}^{\mathrm{R}}_j D^{\Lambda *}_{p',p''} = -\left(\mathsf{H}^{\Lambda}_k\right)_{p'',p''} D^{\Lambda *}_{p',p''} = -(\lambda - p'') D^{\Lambda *}_{p',p''} \tag{29}$$

$$\check{E}^{\mathrm{L}}_\alpha D^{\Lambda *}_{p',p''} = +\sum_p \left(\mathsf{E}^{\Lambda}_\alpha\right)_{p,p'} D^{\Lambda *}_{p,p''}$$

$$\check{E}^{\mathrm{R}}_\alpha D^{\Lambda *}_{p',p''} = -\sum_p \left(\mathsf{E}^{\Lambda}_\alpha\right)_{p'',p} D^{\Lambda *}_{p',p}. \tag{30}$$

(iv) The functions (25), both under the operators $\check{U}^{\mathrm{L}}$ and $\check{U}^{\mathrm{R}}$, transform according to a unitary irreducible matrix representation (IRREP) of $\mathcal{G}$:

$$\check{U}^{\mathrm{L}}(g)D^{\Lambda *}_{p',p''} = \sum_{p}\left(\mathbf{D}^{\Lambda}(g)\right)_{p,p'} D^{\Lambda *}_{p,p''}$$

$$\check{U}^{\mathrm{R}}(g)D^{\Lambda *}_{p',p''} = \sum_{p}\left(\mathbf{D}^{\Lambda}(g)\right)^{*}_{p,p''} D^{\Lambda *}_{p',p}. \tag{31}$$

The explicit form of the matrices $\mathbf{D}^{\Lambda}$ depends on the parametrization of the group elements g.

$$\left(\mathbf{D}^{\Lambda}(g)\right)^{*}_{p',p''} = D^{\Lambda *}_{p',p''}(\eta) \qquad \text{for } g = g(\eta). \tag{32}$$

Up to now the parameters η were only assumed to be global coordinates. We now assume that these coordinates are adapted to the Cartan decomposition (6), (7) of the Lie algebra $\mathbf{g}$. Let $\mathcal{H}$ be the Abelian subgroup of $\mathcal{G}$ generated by the commuting self-adjoint operators $\check{H}^{\mathrm{R}}_1, \ldots, \check{H}^{\mathrm{R}}_r$ and $\eta^H = \eta^H_1, \ldots, \eta^H_r$ be the corresponding canonical coordinates. Then the coordinates of the unit element of $\mathcal{H}$ are $\eta^H_0 = 0$ and

$$\check{H}^{\mathrm{R}}_j = \mathrm{i}\frac{\partial}{\partial \eta^H_j}. \tag{33}$$

The remaining coordinates η^C label the coset representatives in the decomposition $\mathcal{G} = \mathcal{C}\mathcal{H}$.

$$g(\eta^C, \eta^H) = g(\eta^C, 0)g(\eta^C_0, \eta^H) \qquad g(\eta^C, 0) \in \mathcal{C} \qquad g(\eta^C_0, \eta^H) \in \mathcal{H} \tag{34}$$

For these coordinates the operators $\check{E}^{\mathrm{R}}_{\alpha}$ are of the form

$$\check{E}^{\mathrm{R}}_{\alpha} = \exp\left(-\mathrm{i}\sum_{j}\alpha_j \eta^H_j\right)\sum_{k}\Psi_{\alpha k}(\eta^C)\frac{\partial}{\partial \eta_k} \tag{35}$$

and

$$D^{\Lambda *}_{p',p''}(\eta^C, \eta^H) = S^{\Lambda}_{p',p''}(\eta^C)\exp\left(\mathrm{i}\sum_{j}(\lambda - p'')_j \eta^H_j\right). \tag{36}$$

3. The product formula

Consider a quantum system whose state space is a carrier space of the IRREP $\mathbf{D}^{\Lambda}$ of the compact semisimple Lie group $\mathcal{G}$.

$$\hat{U}(g)\psi(p) = \sum_{p'}\left(\mathbf{D}^{\Lambda}(g)\right)_{p',p}\psi(p')$$

$$\hat{K}_k\psi(p) = \sum_{p'}\left(\mathbf{K}^{\Lambda}_k\right)_{p',p}\psi(p'). \tag{37}$$

The quantum mechanical operators $\hat{K}_k = \hat{K}_k^\dagger$ satisfy the same commutation relations as the operators $\check{K}_k^L$ acting in $L^2(\mathcal{G}, \eta)$. Therefore their complex linear combinations contain operators $\hat{H}_i$, $\hat{E}_\alpha$ that satisfy also equations (6), (7). Their action on the states $\psi(p)$ is given by

$$\hat{H}_j \psi(p) = (\lambda - p)_j \psi(p) \qquad \hat{E}_\alpha \psi(p) = \sum_{p'} \left(\mathbf{E}_\alpha^\Lambda\right)_{p',p} \psi(p') \tag{38}$$

whence

$$\hat{E}_{+\alpha}\psi(0) = 0 \qquad \omega(\lambda, p)^*(\lambda \mid p)^{-1/2}\hat{E}_{-p}\psi(0) = \psi(p). \tag{39}$$

Coherent states are introduced by the definition

$$\phi(\eta) = \hat{U}(\eta)\psi(p_0) \tag{40}$$

where for the present p_0 is some fixed pattern belonging to the set $\{p\}$; for reasons that will be discussed below p_0 will be chosen later on to be the pattern $p = 0$ corresponding to the maximal weight [16]. The function A assigned to the operator $\hat{A}$, the so-called Q representative of the operator, is given by

$$A(\eta) = \langle \phi(\eta), \hat{A}\phi(\eta)\rangle = \langle \psi(p_0), \hat{U}(\eta)^\dagger \hat{A}\hat{U}(\eta)\psi(p_0)\rangle. \tag{41}$$

It follows from (41) that the Q representative of $A^\dagger$ is the function A^*. As a consequence of the irreducibility of $\mathbf{D}^\Lambda$

$$\mathrm{Tr}\,\hat{A} = n_\Lambda \int A(\eta)\mu(\eta)\,\mathrm{d}\eta. \tag{42}$$

The operators of the system form a linear space of dimension n_Λ^2 which becomes a unitary space if a scalar product is introduced by the definition

$$\langle \hat{A},\, \hat{B}\rangle = \mathrm{Tr}\,(\hat{A}^\dagger \hat{B}). \tag{43}$$

In this space the shift and projection operators $\hat{E}^\Lambda_{p,p'}$

$$\hat{E}^\Lambda_{p,p''}\psi(p') = \delta_{p'',p'}\psi(p) \tag{44}$$

obviously form a basis. Another basis, that is also orthonormalized in the sense of (43), is given by the operators [25]

$$\hat{Z}^\Lambda_{K,q,v} = \sum_{p',p''} [\Lambda p' \bar{\Lambda} p'' \mid K q v] \hat{E}^\Lambda_{p',p''} \tag{45}$$

where the coefficients of the operators $\hat{E}^\Lambda_{p',p''}$ are the elements of a unitary matrix that decomposes the reducible representation $\mathbf{D}^\Lambda \times \mathbf{D}^{\Lambda *} \approx \mathbf{D}^\Lambda \times \mathbf{D}^{\bar{\Lambda}} \approx \oplus m(\Lambda, \bar{\Lambda}; K)\mathbf{D}^K$

$$\left(\mathbf{D}^\Lambda(g)\right)_{p,p''} \left(\mathbf{D}^{\Lambda *}(g)\right)_{p',p'''} = \sum_{K,q,q',v} [\Lambda p \bar{\Lambda} p' \mid K q v] \left(\mathbf{D}^K(g)\right)_{q,q'} [\Lambda p'' \bar{\Lambda} p''' \mid K q' v]^*. \tag{46}$$

In (46) the range of the multiplicity index v is

$$v = 1, \ldots, m(\Lambda, \bar{\Lambda}; K) \le n_\Lambda. \tag{47}$$

The n_Λ^2 operators (45) are irreducible tensor operators, i.e. they transform according IRREPs under the mapping $\hat{A} \to \hat{U}(g)\,\hat{A}\hat{U}(g)^\dagger$.

$$\hat{U}(g)\hat{Z}^\Lambda_{K,q,v}\hat{U}(g)^\dagger = \sum_{q'} \left(\mathbf{D}^K(g)\right)_{q',q} \hat{Z}^\Lambda_{K,q',v}$$

$$\left[\hat{K}_k, \hat{Z}^\Lambda_{K,q,v}\right] = \sum_{q'} \left(\mathbf{K}^K_k\right)_{q',q} \hat{Z}^\Lambda_{K,q',v}. \tag{48}$$

The Q representatives of the operators (45) are the functions

$$Z^\Lambda_{K,q,v}(\eta) = \sum_{q'} D^{K*}_{q,q'}(\eta)[\Lambda p_0 \bar{\Lambda} p_0 \mid K q' v] \tag{49}$$

which are orthogonal in K and q because of (27). Starting from a given set of coupling coefficients $[\Lambda p \bar{\Lambda} p' \mid K q v]$ it is always possible to pass to new coupling coefficients by means of a unitary transformation in the multiplicity index such that the functions (49) also become orthogonal in the index v.

$$\langle Z^\Lambda_{K,q,v}, Z^\Lambda_{K',q',v'} \rangle = \delta_{K,K'}\delta_{q,q'}\delta_{v,v'} \sum_{q''} \|[\Lambda p_0 \bar{\Lambda} p_0 \mid K q'' v]\|^2. \tag{50}$$

If for some $K = K'$ the sum on the right-hand side of (50) vanishes the Q representative of this tensor operator is equal to zero. As has been shown in [26, 27] this cannot happen if p_0 refers to a maximal weight. This is the reason for choosing $p_0 = 0$ in the following, i.e. to derive the coherent states from the state of highest weight [16]. The n_Λ^2 tensor operators (45) are then uniquely related to the n_Λ^2 functions (49) which span a subspace of $L^2(\mathcal{G}, \eta)$. It is obvious that this entails a one-to-one correspondence between operators and functions in this subspace.

If the relation between operators and functions is one-to-one the Q representative of the product of two operators is uniquely determined by the functions corresponding to the two operators and by the order of the multiplication.

$$\hat{A}\hat{B} = \hat{C} \Longleftrightarrow C = A \circ B. \tag{51}$$

The derivation of a formula for the 'product' $A \circ B$ is based on the following sequence of equations:

$$\begin{aligned}
&\langle \psi(p), \hat{U}(\eta)^\dagger \hat{Z}^\Lambda_{K,q,v}\hat{U}(\eta)\psi(0)\rangle \\
&\quad = \langle \omega(\lambda,p)(\lambda,p)^{-1/2}\hat{E}_{-p}\psi(0), \sum_{q'} D^K_{q',q}(\eta^{-1})\hat{Z}^\Lambda_{K,q',v}\psi(0)\rangle \\
&\quad = \omega(\lambda,p)^*(\lambda,p)^{-1/2}\sum_{q'}\langle\psi(0), \hat{E}_{+\gamma}\ldots\hat{E}_{+\alpha}\hat{Z}^\Lambda_{K,q',v}\psi(0)\rangle D^{K*}_{q,q'}(\eta) \\
&\quad = \omega(\lambda,p)^*(\lambda,p)^{-1/2}\sum_{q',q''}\langle\psi(0), \hat{Z}^\Lambda_{K,q'',v}\psi(0)\rangle \left(\mathbf{E}^K_{+\gamma}\ldots\mathbf{E}^K_{+\alpha}\right)_{q'',q'} D^{K*}_{q,q'}(\eta) \\
&\quad = \omega(\lambda,p)^*(\lambda,p)^{-1/2}\check{E}^{\mathrm{R}}_{+p}\sum_{q''}\langle\psi(0), \hat{Z}^\Lambda_{K,q'',v}\psi(0)\rangle D^{K*}_{q,q''}(\eta).
\end{aligned} \tag{52}$$

In the first of these equations (39) and (48) is used; the second equation follows from relationships analogous to (6); the third from (48) and (39); and in the last equation (30) and (19) have been used. Since (52) may be extended to any linear combination of the tensor operators (45)

$$\langle\psi(p), \hat{U}(\eta)^\dagger \hat{B}\hat{U}(\eta)\psi(0)\rangle = \omega(\lambda, p)^*(\lambda, p)^{-1/2}[\check{E}^{\mathrm{R}}_{+p}B](\eta) \tag{53}$$

for all operators $\hat{B}$. The complex conjugate of (53) is

$$\langle\psi(0), \hat{U}(\eta)^\dagger \hat{A}\hat{U}(\eta)\psi(p)\rangle = \omega(\lambda, p)(\lambda, p)^{-1/2}[\check{E}^{\mathrm{R}*}_{+p}A](\eta) \tag{54}$$

if $\hat{B}^\dagger = \hat{A}$. Multiplying (54) with (53) and summing over all patterns one finally obtains the product formula

$$A \circ B = \sum_p (\lambda, p)^{-1}[\check{E}^{\mathrm{R}*}_{+p}A][\check{E}^{\mathrm{R}}_{+p}B]. \tag{55}$$

It should be noted that relation (53), from which the product formula (55) follows, implies a number of differential equations the functions $D^{\Lambda*}_{p',p''}$ have to satisfy in addition to those that follow directly from their definition (25). To see this consider the function

$$\begin{aligned} Q^{\Lambda}_{p',p'';p}(\eta) &= D^{\Lambda*}_{p',p}(\eta) D^{\Lambda}_{p'',0}(\eta) \\ &= \langle\psi(p), \hat{U}(\eta)^\dagger \hat{E}^{\Lambda}_{p',p''}\hat{U}(\eta)\psi(0)\rangle \end{aligned} \tag{56}$$

where the operator $\hat{E}^{\Lambda}_{p',p''}$ is defined in (44). If in (53) $\hat{B}$ is chosen to be this operator one obtains

$$Q^{\Lambda}_{p',p'';p} = \omega(\lambda, p)^*(\lambda \mid p)^{-1/2}\check{E}^{\mathrm{R}}_{+p} Q^{\Lambda}_{p',p'';0} \tag{57}$$

which is equivalent to the following two sets of equations:

$$\omega(\lambda, p)^*(\lambda \mid p)^{-1/2}\, \check{E}^{\mathrm{R}}_{+p} D^{\Lambda*}_{p',0} = D^{\Lambda*}_{p',p} \tag{58}$$

$$\check{E}^{\mathrm{R}*}_{+p} D^{\Lambda*}_{p',0} = 0. \tag{59}$$

Whereas (58) is contained in (25) the differential equations (59) constitute equations of a form not met up to now; they contain the operators $\check{E}^{\mathrm{R}*}_{+p}$ whose properties have not yet been discussed. In the example discussed in section 4 relations (59) follow from equations (15) that were also used in the present derivation of (55) in form of the first of equations (39). But irrespective of whether (59) is derived via the introduction of the tensor operators (45), or in some other way, the full set of equations (58, 59) suffices to derive the product formula (55). This follows from the fact that the functions $Q^{\Lambda}_{p',p'';0}$ are nothing but the Q representatives of the operators $\hat{E}^{\Lambda}_{p',p''}$ in terms of which any other operators may be represented.

There exist many global coordinates η that may be used to define coherent states and the corresponding Q representatives. However, the most economic way to relate operators to functions is to use coordinates adapted to the Cartan decomposition of the Lie algebra. In this case the Q representatives vary only over the coset coordinates η^C as is easily seen from (40),(37), and (36).

$$A(\eta^C, \eta^H) = A^{(C)}(\eta^C). \tag{60}$$

Since this relation holds also for the Q representatives of the tensor operators (45) no pattern q' with weight $\kappa - q' \neq 0$ can contribute to the sum (49); this corresponds to a selection rule for the coupling coefficients in the decomposition (46).

The differential operators (35) occurring in the product formula (55) depend on all coordinates η_k but the dependence on the coordinates η_j^H drops out in each term of (55).

$$\begin{aligned}
\check{E}^{\mathrm{R}}_{+p} B^{(C)} &= \left(-\check{E}^{\mathrm{R}}_{\gamma}\right) \dots \left(-\check{E}^{\mathrm{R}}_{\beta}\right) \left(-\check{E}^{\mathrm{R}}_{\alpha}\right) B^{(C)} \\
&= \exp\left(-\mathrm{i} \sum_j (+p)_j \eta_j^H\right) \check{E}^{\mathrm{R}(C)}_{\gamma, +p-\gamma} \dots \check{E}^{\mathrm{R}(C)}_{\beta,\alpha} \check{E}^{\mathrm{R}(C)}_{\alpha,0} B^{(C)} \\
&= \exp\left(-\mathrm{i} \sum_j (+p)_j \eta_j^H\right) \check{E}^{\mathrm{R}(C)}_{+p} B^{(C)}
\end{aligned} \tag{61}$$

$$\check{E}^{\mathrm{R}(C)}_{\delta,\zeta} = \sum_{k \neq j} \Psi_{\delta k}(\eta^C) \frac{\partial}{\partial \eta_k^C} - \mathrm{i} \sum_j \Psi_{\delta j}(\eta^C) \zeta_j. \tag{62}$$

With this definition of the differential operators $\check{E}^{\mathrm{R}(C)}_{+p}$ the product formula for the Q representatives (60) reads

$$A^{(C)} \circ B^{(C)} = \sum_p (\lambda, p)^{-1} \left[\check{E}^{\mathrm{R}(C)*}_{+p} A^{(C)}\right] \left[\check{E}^{\mathrm{R}(C)}_{+p} B^{(C)}\right]. \tag{63}$$

We close this section with a comment on the 'classical' or 'large N' limit. Here one considers a sequence of representations where

$$(\lambda \mid \alpha) = \sum_j \lambda_j \alpha_j = \mathrm{O}(N^{-1}) \tag{64}$$

and N ($\propto \hbar^{-1}$) tends to infinity. In this limit the value of the function $A \circ B$ at position η approaches the product $A(\eta)B(\eta)$, the largest 'quantum corrections' being of order N^{-1} (cf (22)).

$$[A \circ B](\eta) = A(\eta)B(\eta) + \sum_{+\alpha \in \{p\}} (\lambda \mid \alpha) \left[\check{E}^{\mathrm{R}*}_{+\alpha} A\right](\eta) \left[\check{E}^{\mathrm{R}}_{+\alpha} B\right](\eta) + \mathrm{O}(N^{-2}). \tag{65}$$

Accordingly the function corresponding to the commutator of two operators vanishes as N^{-1}. However, the Q representative of the scaled commutator approaches a finite limit which defines the 'Poisson bracket' of the classical theory.

$$\begin{aligned}
N[A \circ B - B \circ A] = \sum_{+\alpha \in \{p\}} & N(\lambda \mid \alpha) \left\{ \left[\check{E}^{\mathrm{R}*}_{+\alpha} A\right] \left[\check{E}^{\mathrm{R}}_{+\alpha} B\right] \right. \\
& \left. - \left[\check{E}^{\mathrm{R}*}_{+\alpha} B\right] \left[\check{E}^{\mathrm{R}}_{+\alpha} A\right] \right\} + \mathrm{O}(N^{-1})
\end{aligned} \tag{66}$$

4. Coherent spin states

In this section the product formula (59) is specified for the group SU(2), i.e. for Q representatives related to coherent spin states. If the group elements are labelled by Euler angles $\omega = (\phi, \theta, \psi)$ then the domain $-\pi < \phi < \pi$, $0 < \theta < \pi$, $-2\pi < \psi < 2\pi$ contains all elements of SU(2) up to a set of measure zero (which includes the unit element $(0,0,0)$). The weight function is $\mu(\omega) = (1/16\pi^2)\sin\theta$ and the analytic functions in $L^2(\mathrm{SU}(2), \omega)$ turn out to be periodic in all three variables, the common period being 4π. Because of this periodicity it is possible to choose the angles $\omega^{-1} = (-\psi, -\theta, -\phi)$ as labels of the element $g(\omega)^{-1}$ although these parameters range over a different domain than the coordinates ω. The subgroup $\mathcal{H}$ of section 3 consists of the rotations about the z-axis and the relation of the Euler angles to the coordinates η of the preceding section is given by

$$\begin{aligned} \eta &= (\eta_1^C, \eta_2^C, \eta^H) = (\phi, \theta, \sqrt{2}\psi) \\ \eta^{-1} &= ((\eta^{-1})_1^C, (\eta^{-1})_2^C, (\eta^{-1})^H) = (-\psi, -\theta, -\sqrt{2}\phi). \end{aligned} \tag{67}$$

The infinitesimal generators of the right- and left-transformations are denoted by $\check{J}$ instead of $\check{K}$ and expressed in terms of the Euler angles to indicate their relation to the angular momentum operators of wave mechanics.

$$\begin{aligned} \check{J}_1^{\mathrm{R}} &= \mathrm{i}\left(-\mathrm{cosec}\theta\cos\psi\frac{\partial}{\partial\phi} + \sin\psi\frac{\partial}{\partial\theta} + \cot\theta\cos\psi\frac{\partial}{\partial\psi}\right) \\ \check{J}_2^{\mathrm{R}} &= \mathrm{i}\left(+\mathrm{cosec}\theta\sin\psi\frac{\partial}{\partial\phi} + \cos\psi\frac{\partial}{\partial\theta} - \cot\theta\sin\psi\frac{\partial}{\partial\psi}\right) \\ \check{J}_3^{\mathrm{R}} &= +\mathrm{i}\frac{\partial}{\partial\psi} \end{aligned} \tag{68}$$

$$\begin{aligned} \check{J}_1^{\mathrm{L}} &= \mathrm{i}\left(+\cos\phi\cot\theta\frac{\partial}{\partial\phi} + \sin\phi\frac{\partial}{\partial\theta} - \cos\phi\,\mathrm{cosec}\theta\frac{\partial}{\partial\psi}\right) \\ \check{J}_2^{\mathrm{L}} &= \mathrm{i}\left(+\sin\phi\cot\theta\frac{\partial}{\partial\phi} - \cos\phi\frac{\partial}{\partial\theta} - \sin\phi\,\mathrm{cosec}\theta\frac{\partial}{\partial\psi}\right) \\ \check{J}_3^{\mathrm{L}} &= -\mathrm{i}\frac{\partial}{\partial\phi}. \end{aligned} \tag{69}$$

The operators (68) and (69) satisfy the same commutation relations as angular momentum operators for $\hbar = 1$ ($[\check{J}_1, \check{J}_2] = \mathrm{i}\check{J}_3$ etc). The Lie algebra is of rank one and the operators

$$\check{H} = \frac{1}{\sqrt{2}}\check{J}_3 \qquad \check{E}_{\pm 1/\sqrt{2}} = \frac{1}{2}(\check{J}_1 \pm \mathrm{i}\check{J}_2) \tag{70}$$

form a basis adapted to the Cartan decomposition. The root vectors are $\pm\alpha = \pm 1/\sqrt{2}$ and the canonical commutation relations are

$$[\check{H}, \check{E}_{\pm 1/\sqrt{2}}] = \pm\frac{1}{\sqrt{2}}\check{E}_{\pm 1/\sqrt{2}} \qquad [\check{E}_{+1/\sqrt{2}}, \check{E}_{-1/\sqrt{2}}] = \frac{1}{\sqrt{2}}\check{H}. \tag{71}$$

In terms of the Euler angles the differential operators (66) and the Casimir operator assume the following form:

$$\check{H}^{\mathrm{R}} = +\frac{\mathrm{i}}{\sqrt{2}}\frac{\partial}{\partial\psi} \qquad \check{E}^{\mathrm{R}}_{\pm 1/\sqrt{2}} = \frac{1}{2}\exp(\mp\mathrm{i}\psi)\left(-\mathrm{i}\,\mathrm{cosec}\theta\frac{\partial}{\partial\phi} \mp \frac{\partial}{\partial\theta} + \mathrm{i}\cot\theta\frac{\partial}{\partial\psi}\right) \tag{72}$$

$$\check{H}^{\mathrm{L}} = -\frac{\mathrm{i}}{\sqrt{2}}\frac{\partial}{\partial\phi} \qquad \check{E}^{\mathrm{L}}_{\pm 1/\sqrt{2}} = \frac{1}{2}\exp(\pm\mathrm{i}\phi)\left(+\mathrm{i}\cot\theta\frac{\partial}{\partial\phi} \pm \frac{\partial}{\partial\theta} - \mathrm{i}\,\mathrm{cosec}\theta\frac{\partial}{\partial\psi}\right) \tag{73}$$

$$\begin{aligned}\check{C} &= \sum_k (\check{J}^{\mathrm{R}}_k)^2 = \sum_k (\check{J}^{\mathrm{L}}_k)^2 = \check{C}^{\#} \\ &= -\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \mathrm{cosec}^2\theta\left(\frac{\partial^2}{\partial\phi^2} - 2\cos\theta\frac{\partial^2}{\partial\phi\partial\psi} + \frac{\partial^2}{\partial\psi^2}\right)\right].\end{aligned} \tag{74}$$

For the present example the solution of the differential equations (13–15), that satisfies the boundary condition (16), is

$$T^{S(S+1)}_{+S/\sqrt{2},-S/\sqrt{2}} = \exp(\mathrm{i}S\phi)\left(\cos\frac{\theta}{2}\right)^{2S}\exp(\mathrm{i}S\psi) \tag{75}$$

where $2S$ is a non-negative integer. The Gel'fand–Tsetlein patterns are labelled by

$$K = 0, \ldots, 2S. \tag{76}$$

The operators $\check{E}^{\mathrm{L}}_{-0}$ and $\check{E}^{\mathrm{R}}_{+0}$ are given by (24), and the shift operators for $K \geq 1$ read

$$\check{E}_{\pm K} = \left(\mp\check{E}_{\pm 1/\sqrt{2}}\right)^K. \tag{77}$$

Using the commutation relations

$$[\check{E}_{+1/\sqrt{2}}, (\check{E}_{-1/\sqrt{2}})^K] = (\check{E}_{-1/\sqrt{2}})^{K-1}\left(\frac{K}{\sqrt{2}}\check{H} - \frac{K(K-1)}{4}\right) \tag{78}$$

that may be derived from (71) by induction, one obtains the normalization constants

$$(S \mid K) = \frac{K!(2S)!}{2^{2K}(2S-K)!}. \tag{79}$$

If all phase factors $\omega(S,K)$ are chosen to be equal to $(+1)$ then the function

$$D^{S(S+1)*}_{K',K''}(\phi,\theta,\psi) = \mathbf{D}^S_{S-K',S-K''}(\phi,\theta,\psi)^* \tag{80}$$

derived from (71) according to (25), is nothing but the complex conjugate element of the usual rotation matrix $\mathbf{D}^S$ [28, 29] at position $M' = S-K', M'' = S-K''$. The explicit form of these matrix elements can be calculated from

$$\mathbf{D}^S_{S,S} = \exp(\mathrm{i}S\phi)\left(\cos\frac{\theta}{2}\right)^{2S}\exp(\mathrm{i}S\psi) \tag{81}$$

and the following recurrence relations:

$$(+2)\check{E}^{\mathrm{L}}_{\pm 1/\sqrt{2}}\mathbf{D}^{S*}_{S-K,S-L} == \sqrt{\left(K+\tfrac{1}{2}\mp\tfrac{1}{2}\right)\left(2S-K+\tfrac{1}{2}\pm\tfrac{1}{2}\right)}\mathbf{D}^{S*}_{S-K\pm 1,S-L} \tag{82}$$

$$(-2)\check{E}^{\mathrm{R}}_{\pm 1/\sqrt{2}}\mathbf{D}^{S*}_{S-K,S-L} == \sqrt{\left(L+\tfrac{1}{2}\pm\tfrac{1}{2}\right)\left(2S-L+\tfrac{1}{2}\mp\tfrac{1}{2}\right)}\mathbf{D}^{S*}_{S-K,S-L\mp 1}. \tag{83}$$

One half of these relations results from definition (25), specialized to $\mathcal{G} = \mathrm{SU}(2)$, while the other half may be deduced from these equations, the commutation relations (67), and the differential equations the function (81) satisfies (cf (77) and (13)–(15)). Because

$$\check{E}^{*}_{+1/\sqrt{2}} = -\check{E}_{-1/\sqrt{2}} \tag{84}$$

equations (83) include condition (59) so that the product formula may be derived from the recurrence equations (82, 83) (cf section 3).

From (72) the shift operators acting on functions of the coset variables ϕ, θ may be derived; one finds $\check{E}^{\mathrm{R}(C)}_{+0} = 1$ and, for $K \geq 1$,

$$\begin{aligned} 2^K \check{E}^{\mathrm{R}(C)}_{+K} = &\left[\mathrm{i}\operatorname{cosec}\theta\frac{\partial}{\partial\phi} + \frac{\partial}{\partial\theta} + (K-1)\cot\theta\right] \\ &\ldots\left[\mathrm{i}\operatorname{cosec}\theta\frac{\partial}{\partial\phi} + \frac{\partial}{\partial\theta} + \cot\theta\right]\left[\mathrm{i}\operatorname{cosec}\theta\frac{\partial}{\partial\phi} + \frac{\partial}{\partial\theta}\right]. \end{aligned} \tag{85}$$

The resulting product formula

$$A^{(C)} \circ B^{(C)} = \sum_K \frac{(2S-K)!}{K!(2S)!}\left[2^K \check{E}^{\mathrm{R}(C)*}_{+K} A^{(C)}\right]\left[2^K \check{E}^{\mathrm{R}(C)}_{+K} B^{(C)}\right] \tag{86}$$

clearly shows that the classical limit is approached for $S \to \infty$.

It is also instructive to consider the tensor operators (45) and their Q representatives (49) in this example. As SU(2) is simply reducible the multiplicity index v may be dropped. It turns out that $\mathbf{D}^{S*} \sim \mathbf{D}^{S}$ and the coupling coefficients are closely related to the Clebsch–Gordan coefficients:

$$[S\,K'\,S\,K''\,|\,L\,L-M] = \delta_{K',K''-M}\,\mathrm{i}^{2(S-K'')}(\,S\ S-K''+M\ S\ -S+K''\,|\,L\ M\,). \tag{87}$$

The Q representatives of the tensor operators are therefore

$$\begin{aligned} Z^S_{LM}(\phi,\theta,\psi) &= \mathrm{i}^{2S}(\,S\ S\ S\ -S\,|\,L\ 0\,)\mathbf{D}^S_{M,0}(\phi,\theta,\psi)^* \\ &= \mathrm{i}^{2S}(\,S\ S\ S\ -S\,|\,L\ 0\,)\left[\frac{4\pi}{2L+1}\right]^{1/2} Y_{LM}(\theta,\phi). \end{aligned} \tag{88}$$

Accordingly the Q representative of any other operator is a linear combination of spherical harmonics Y_{LM}, the integer L ranging from 0 to $2S$. The one-to-one correspondence between spherical harmonics and tensor operators allows one to assign to each phase-space function a linear combination of tensor operators where these operators may be represented as polynomials in the angular momentum operators $\hat{J}_k$ (see [25] and the references cited therein).

5. Coherent oscillator states

It is well known that the compact group U(2) may be contracted into the non-compact group $\mathcal{H}_4$, where a subset of the rotation operators is transformed into unitary operators that multiply like the Weyl operators $\hat{T}(P, Q)$ (see, e.g., [20, 30]). Here this relation is used to derive a product formula for the Q representatives that are obtained from the 'canonical' coherent states studied by Schrödinger, Glauber and others [9].

To see the relation of these coherent states to the coherent spin states of the previous section consider first the functions

$$\mathbf{D}^{S*}_{S-N,0}(\alpha,\beta,\gamma) = \left[\frac{4\pi}{2S+1}\right]^{1/2} Y_{S,S-N}(\beta,\alpha)$$
$$= P_S^{S-N}(\cos\beta)\exp[\mathrm{i}(S-N)\alpha] = (-1)^S \exp(\mathrm{i}S\alpha) F_N^S(\alpha,\beta). \tag{89}$$

For large S and small θ the associated Legendre functions P_S^{S-N} are similar in form to the oscillator eigenfunctions

$$\chi_N(Z) = \left[\sqrt{\pi}\,2^N N!\right]^{-1/2}\left(Z - \frac{\partial}{\partial Z}\right)^N \exp(\tfrac{1}{2}Z^2) \tag{90}$$

for $Z > 0$. In the limit $S \to \infty$ this similarity becomes a quantitative relation if the variable β is scaled according to

$$\beta = \frac{\pi}{2} - \frac{Z}{\sqrt{S}} \qquad (S \to \infty) \tag{91}$$

$$F_N^S(\alpha,\beta) \to F_N(\alpha, Z) = (-1)^N \exp(-\mathrm{i}N\alpha)\chi_N(Z). \tag{92}$$

Relation (92) is easily verified for $N = 0$; the remaining limits emerge from the recursion relations (82). If the operators $\hat{A}^S_\mp$ are defined as

$$\hat{A}^S_\mp = \exp(-\mathrm{i}S\alpha)\sqrt{\frac{2}{S}}\,\hat{E}_{\pm 1/\sqrt{2}}\exp(\mathrm{i}S\alpha) \tag{93}$$

equations (82) give

$$\hat{A}^S_\pm F_N^S = \sqrt{N + \tfrac{1}{2} \pm \tfrac{1}{2}}\,F^S_{N\pm 1} + \mathrm{O}(S^{-1}). \tag{94}$$

Here the shift operators for the functions (89) are

$$\hat{E}_{\pm 1/\sqrt{2}} = \exp(\pm\mathrm{i}\alpha)\left(\mathrm{i}\cot\beta\frac{\partial}{\partial\alpha} \pm \frac{\partial}{\partial\beta}\right) \tag{95}$$

(cf (73)), so that

$$\hat{A}^S_\pm = -\exp(\mp\mathrm{i}\alpha)\frac{1}{\sqrt{2}}\left(Z \mp \frac{\partial}{\partial Z}\right) + \mathrm{O}(S^{-1}) \tag{96}$$

and (92) follows from (94) and (90).

Under the rotation operators the functions (89) transform according to the IRREP $\mathbf{D}^S$ of SU(2) (cf (31))

$$\hat{U}(\phi,\theta,\psi)\mathbf{D}^{S*}_{S-N,0} = \sum_M \mathbf{D}^S_{S-M,S-N}(\phi,\theta,\psi)\mathbf{D}^{S*}_{S-M}. \tag{97}$$

This equation may be transcribed as

$$\hat{R}^S(\phi,\theta,\psi)F^S_N = \sum_M \exp[-\mathrm{i}S(\phi+\psi)]\Delta^S_{M,N}(\phi,\theta,\psi)F^S_M \tag{98}$$

where

$$\hat{R}^S(\phi,\theta,\psi) = \exp(-\mathrm{i}S\alpha)\hat{U}(\phi,\theta,\psi)\exp(\mathrm{i}S\alpha) \tag{99}$$

and

$$\Delta^S_{M,N}(\phi,\theta,\psi) = \exp[\mathrm{i}S(\phi+\psi)]\mathbf{D}^S_{S-M,S-N}(\phi,\theta,\psi). \tag{100}$$

As we are interested in the form of the functions F^S_N near the equator the rotation angle θ has to be restricted to extremely small values.

$$\theta = \frac{W}{\sqrt{S}} \qquad (S\to\infty). \tag{101}$$

In addition we set $\psi = -\phi$ since the value of the third angle is irrelevant for the Q representatives. The remaining unitary transformations are then of the form

$$\hat{R}^S(\phi,\frac{W}{\sqrt{S}},-\phi) = \exp\left\{\mathrm{i}\left(P\hat{J}^S_1 - Q\hat{J}^S_2\right)\right\} = T^S(P,Q). \tag{102}$$

Here the parameters P, Q are related to the parameters ϕ, W by

$$Q \pm \mathrm{i}P = W\exp(\pm\mathrm{i}\phi). \tag{103}$$

The infinitesimal generators of the transformations (102) are

$$\hat{J}^S_1 \pm \mathrm{i}\hat{J}^S_2 = \exp(-\mathrm{i}S\alpha)\frac{1}{\sqrt{S}}\left(\hat{J}_1 \pm \mathrm{i}\hat{J}_2\right)\exp(\mathrm{i}S\alpha) = \sqrt{2}\hat{A}^S_{\mp} \tag{104}$$

and satisfy the commutation relations

$$[\hat{J}^S_1, \hat{J}^S_2] = \mathrm{i}\hat{1} + \mathrm{O}(S^{-1}). \tag{105}$$

From these definitions follows the multiplication law

$$\hat{T}^S(P_2,Q_2)\hat{T}^S(P_1,Q_1) = \exp(\mathrm{i}R)\hat{T}^S(P,Q) + \mathrm{O}(S^{-1})$$

$$P_2 + P_1 = P \qquad Q_2 + Q_1 = Q \qquad P_2Q_1 - Q_2P_1 = 2R \tag{106}$$

which approaches the multiplication law of the Heisenberg–Weyl group in the limit $S\to\infty$. The corresponding limits of the generators,

$$\hat{J}^S_1 \to \hat{Q} \qquad \hat{J}^S_2 \to \hat{P} \qquad \hat{A}^S_+ \to \hat{A}^\dagger \qquad \hat{A}^S_- \to \hat{A} \tag{107}$$

satisfy the same commutation relations as the position and momentum operators ($\hbar = 1$) and the creation and destruction operators of the harmonic oscillator, respectively. Therefore the properties of coherent oscillator states may be obtained from those of the coherent spin states although the limits of the basis functions and operators contain a second variable that does not occur in the wave mechanics of the harmonic oscillator.

A matrix representation of the Weyl operators $\hat{T}(P, Q)$, the limits of the operators (102), is obtained from the functions (100) for $S \to \infty$:

$$\Delta^S_{M,N}(\phi, \theta, \psi) \to \Delta_{M,N}(\phi, W, \psi). \tag{108}$$

These functions may be calculated from

$$\Delta^*_{0,0}(\phi, W, \psi) = \exp(-\tfrac{1}{4}W^2) \tag{109}$$

and the recursion relations that emerge from (82, 83):

$$+\check{E}^{\mathrm{L}}_{\pm}\Delta^*_{M,N} = \sqrt{M + \tfrac{1}{2} \mp \tfrac{1}{2}}\Delta^*_{M\mp1,N} \tag{110}$$

$$-\check{E}^{\mathrm{R}}_{\pm}\Delta^*_{M,N} = \sqrt{N + \tfrac{1}{2} \pm \tfrac{1}{2}}\Delta^*_{M,N\pm1}. \tag{111}$$

The shift operators occurring in these equations are defined by

$$\exp[-\mathrm{i}S(\phi + \psi)]\sqrt{\frac{2}{S}}\check{E}_{\pm1/\sqrt{2}}\exp[\mathrm{i}S(\phi + \psi)] \to \check{E}_{\pm} \tag{112}$$

and therefore assume the following form:

$$\begin{aligned}\check{E}^{\mathrm{L}}_{\pm} &= \frac{1}{\sqrt{2}}\mathrm{e}^{\pm\phi}\left[\left(\frac{W}{2} \pm \frac{\partial}{\partial W}\right) + \frac{\mathrm{i}}{W}\left(\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\psi}\right)\right]\\ \check{E}^{\mathrm{R}}_{\pm} &= \frac{1}{\sqrt{2}}\mathrm{e}^{\mp\psi}\left[\left(\frac{W}{2} \mp \frac{\partial}{\partial W}\right) - \frac{\mathrm{i}}{W}\left(\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\psi}\right)\right].\end{aligned} \tag{113}$$

The solution of (109–111) is

$$\Delta^*_{M,N} = \left[2^{M+N}M!N!\right]^{-1/2}\exp\left(-\mathrm{i}M\phi - \mathrm{i}N\psi - \tfrac{1}{4}W^2\right)P_{M,N}(W) \tag{114}$$

with

$$P_{M,N}(W) = (-1)^N\sum_L(-1)^{\mathrm{L}}\frac{2^{\mathrm{L}}M!N!}{L!(M-L)!(N-L)!}W^{M+N-2L}. \tag{115}$$

If $\psi = -\phi$, and ϕ and W are expressed in terms of P and Q (see (103)), the matrices $\Delta(P, Q)$ constitute a projective IRREP of the Heisenberg–Weyl group; this representation coincides with the one obtained by a calculation where conventional Weyl operators and oscillator eigenfunctions have been used (cf [31] p 201).

For SU(2) the product formula can be derived from a recursion relation that relates the functions

$$\begin{aligned} Q^S_{N,M;K}(\phi,\theta,\psi) &= \mathbf{D}^{S*}_{S-N,S-K}(\phi,\theta,\psi)\mathbf{D}^S_{S-M,S}(\phi,\theta,\psi) \\ &= \Delta^{S*}_{N,K}(\phi,\theta,\psi)\Delta^S_{N,K}(\phi,\theta,\psi) \end{aligned} \tag{116}$$

with $K > 0$ to the functions

$$\begin{aligned} Q^S_{N,M}(\phi,\theta) &= Q^S_{N,M;0}(\phi,\theta,\psi) \\ &= \langle F^S_0, \hat{U}(\phi,\theta,\psi)^\dagger \hat{E}^S_{M,N}\hat{U}(\phi,\theta,\psi)F^S_0\rangle \end{aligned} \tag{117}$$

(cf (56), (57)). The functions (117) are the Q representatives of the shift and projection operators $\hat{E}^S_{M,N}$ which form a basis for the operators in spin space. Studying the limits of the functions (117) we implicitly focus on finite linear combinations of the oscillator shift and projection operators $\hat{E}_{M,N}$; more comprehensive classes of operators can be introduced by considering sequences of such operators. The Q representatives (117) are related through the recursion equations

$$\begin{aligned} 2\check{E}^{\mathrm{L}}_{+1/\sqrt{2}}Q^S_{M,N} &= \sqrt{M(2S-M+1)}Q^S_{M-1,N} - \sqrt{(N+1)(2S-N)}Q^S_{M,N+1} \\ 2\check{E}^{\mathrm{L}}_{-1/\sqrt{2}}Q^S_{M,N} &= \sqrt{(M+1)(2S-M)}Q^S_{M+1,N} - \sqrt{N(2S-N+1)}Q^S_{M,N+1} \end{aligned} \tag{118}$$

which follow from (82, 84). In the limit (101) these equations assume the form

$$\hat{A}^{\mathrm{L}}_{\pm}Q_{M,N} = \sqrt{M+\tfrac{1}{2}\mp\tfrac{1}{2}}Q_{M\mp1,N} - \sqrt{N+\tfrac{1}{2}\pm\tfrac{1}{2}}Q_{M,N\pm1} \tag{119}$$

where the differential operators $\hat{A}^{\mathrm{L}}_{\pm}$ are defined by the following limits.

$$\sqrt{\frac{2}{S}}\check{E}^{\mathrm{L}}_{\pm1/\sqrt{2}} \to \check{A}^{\mathrm{L}}_{\pm} = \frac{1}{\sqrt{2}}\exp(\pm\mathrm{i}\phi)\left(\pm\frac{\partial}{\partial W} + \frac{\mathrm{i}}{W}\frac{\partial}{\partial\phi}\right). \tag{120}$$

Equation (119) allows one to calculate the functions $Q_{M,N}$ recursively from $Q_{0,0}$.

$$Q_{M,N}(\phi,W) = [2^{M+N}M!N!]^{-1/2}W^{M+N}\exp[-\mathrm{i}(M-N)\phi - \tfrac{1}{2}W^2]. \tag{121}$$

The same result is obtained from (116) and (114), and (103) may be used to express $Q_{M,N}$ as a function of P and Q. A general Q representative, that belongs to the class under consideration, is therefore the product of a polynomial in the variables P, Q and the exponential $\exp\{-(P^2+Q^2)/2\}$.

The functions (116) may be calculated from the functions (117) because

$$-2\check{E}^{\mathrm{R}}_{+1/\sqrt{2}}Q^S_{M,N;K} = \sqrt{(K+1)(2S-K)}Q^S_{M,N;K+1}$$

this relation follows also from (82–84). In the limit (101)

$$\sqrt{\frac{2}{S}}\check{E}^{\mathrm{R}}_{\pm1/\sqrt{2}} \to \check{A}^{\mathrm{R}}_{\pm} = \frac{1}{\sqrt{2}}\exp(\mp\mathrm{i}\psi)\left[\mp\frac{\partial}{\partial W} - \frac{\mathrm{i}}{W}\left(\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\psi}\right)\right] \tag{123}$$

and

$$(-\check{A}^{\mathrm{R}}_{+})^{K} Q_{M,N} = \sqrt{K!}\, Q_{M,N;K}. \tag{124}$$

As this equation corresponds to relation (54) the product formula for the Q representatives considered here follows immediately. Using the notation

$$(-\check{A}^{\mathrm{R}}_{+})^{K} = \exp(-\mathrm{i}K\psi)\frac{1}{\sqrt{2^K}}\check{D}_K \tag{125}$$

$$\begin{aligned} \check{D}_0 &= \check{1} \\ \check{D}_K &= \left[\frac{\partial}{\partial W} + \frac{1}{W}\left(\mathrm{i}\frac{\partial}{\partial \phi} - K + 1\right)\right]\check{D}_{K-1} \\ &= \frac{1}{\sqrt{P^2+Q^2}}\left[(P+\mathrm{i}Q)\frac{\partial}{\partial P} + (Q-\mathrm{i}P)\frac{\partial}{\partial Q} - K + 1\right]\check{D}_{K-1} \end{aligned} \tag{126}$$

one finally obtains

$$A \circ B = \sum_K \frac{1}{2^K K!}[\check{D}^*_K A][\check{D}_K B]. \tag{127}$$

The relation to classical mechanics becomes more transparent if the variables

$$p = \sqrt{\hbar}P \qquad q = \sqrt{\hbar}Q$$

are used instead of the variables P, Q. The Q representatives of position and momentum operators are then the functions q and p and the Kth term of (127) is proportional to the Kth power of $\hbar$. Moreover

$$A \circ B - B \circ A = \mathrm{i}\hbar\left(\frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}\right) + O(\hbar^2) \tag{129}$$

as is required by the correspondence principle.

6. Conclusion

In 'quasiclassical' or 'phase space' formulations of quantum mechanics operators are replaced by functions. To obtain a fully equivalent description of quantum systems all mathematical operations that are essential for obtaining a result in standard quantum mechanics have to have counterparts in the non-standard formulation. The standard operations include the trace of an operator, the relation between an operator and its adjoint, linear combinations, and products of operators. Whereas the equivalent of all these operations is well known in the Wigner–Weyl formalism, product formulae for functions related to coherent states (Q representatives) have not been discussed in full generality up to now.

In this paper a product formula is first derived in general form for Q representatives related to an arbitrary compact Lie group. This formula is then specified for coherent spin states. Because of the close connection between these states and the canonical coherent states studied by Glauber and others a similar product formula can also be derived for this type of expectation values. Up to now the lack of such a formula prevented the phase space description based on coherent states to become a true alternative to the Wigner–Weyl formalism. Now it is possible to compare these two schemes and to investigate whether in a given problem one is preferable to the other.

The mere reformulation of a problem rarely helps to solve it. However, it has been shown in many instances that equivalent but formally different descriptions of one and the same problem lead to different approximation schemes, or help to elucidate its relation to other problems. It is to be expected that the formulae derived here are of use in a systematic calculation of quantum corrections to classical results. As these formulae relate Q representatives of more complicated operators to those of simpler ones, they should also simplify the calculation of expectation values for coherent states.

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